Signal Analysis & Communication ECE 355

Ch. 4.1: Continuous Time Fourier Transform

Lecture 18

19-10-2023

Ch. 4.1: Continuous Time Fourier Transform (CTFT)

Introduction

Periodic Signals $\xrightarrow{\text{Represented as}}$ $x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

- This representation can be used in describing the effect of LTI system on signal.

$$e^{st} \longrightarrow \boxed{h_{(t)}} \longrightarrow H_{(s)} e^{st}$$

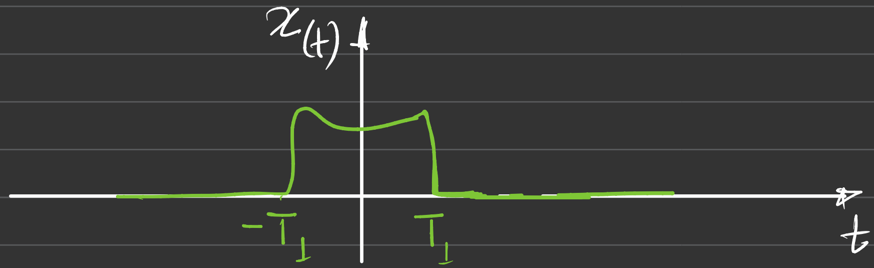
LPF & HPF example.

- Let's extend these concepts to apply to signals that are aperiodic.

- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesmally close in freq., & the representation in terms of linear combination takes the form of an integral rather a sum.

- The resulting spectrum of coefficients in this representation is called the "Fourier Transform".

- Consider aperiodic signal $x_{(t)}$



- From this aperiodic sig., we can construct a periodic sig. $\tilde{x}_{(t)}$ for which $x_{(t)}$ is one period.

$$\tilde{x}(t) = x(t) \quad \text{for} \quad -T/2 \leq t \leq T/2$$

- Fourier Series representation of this sig. is:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad — ①$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

Since $x(t) = \tilde{x}(t)$, for $|t| < T/2$ & $x(t) = 0$, o/w

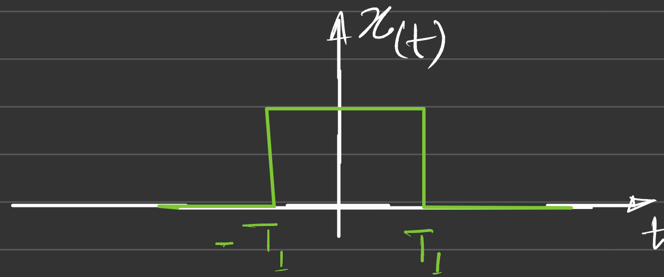
$$Ta_k = \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= X(jk\omega_0) \quad — ②$$

- Define $\underbrace{\text{envelope}}_{k\omega_0 = \omega}$ of Ta_k as $X(j\omega)$ — See Figure below!

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad — ③$$

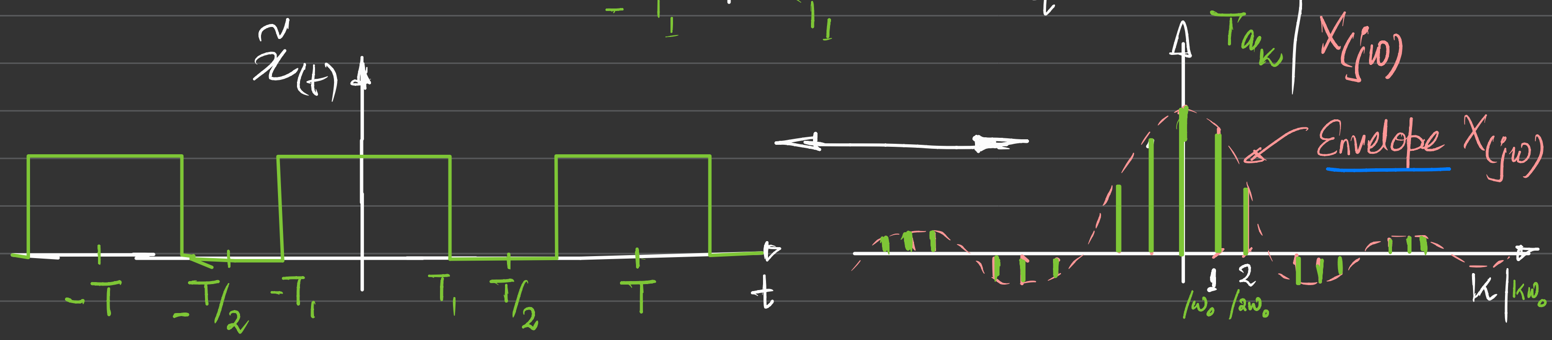
For example:



Envelope intuition

as $\omega_0 \to 0$ $(T \to \infty)$

$k\omega_0 \to \omega$ continuous



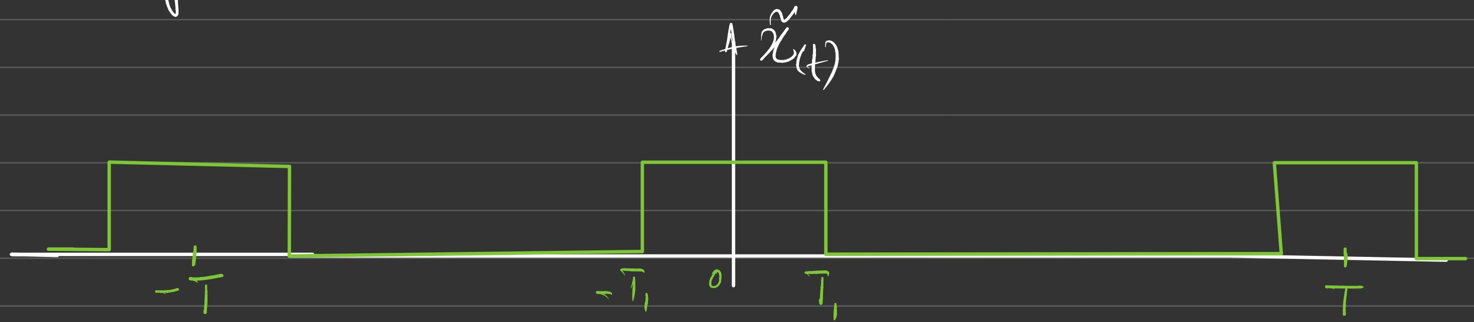
- Combining eqn ① & ②, we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

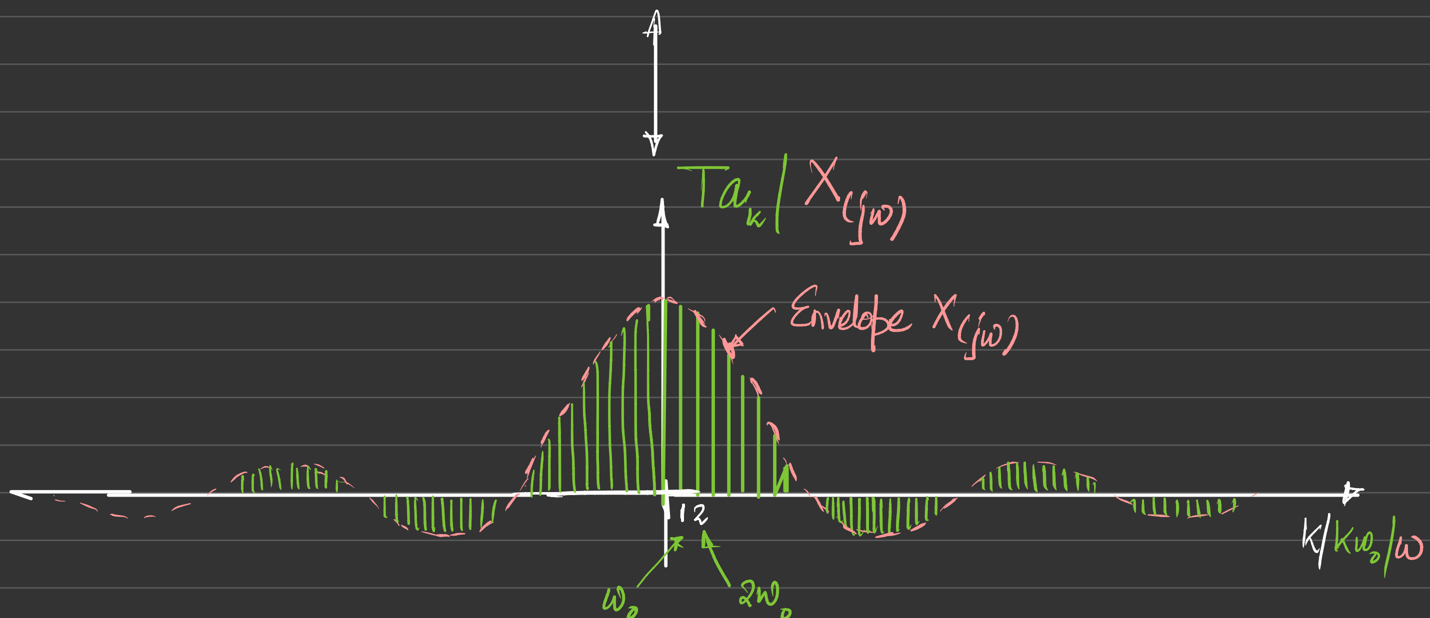
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$$

Since $\omega_0 = 2\pi/T$ or $T = 2\pi/\omega_0$

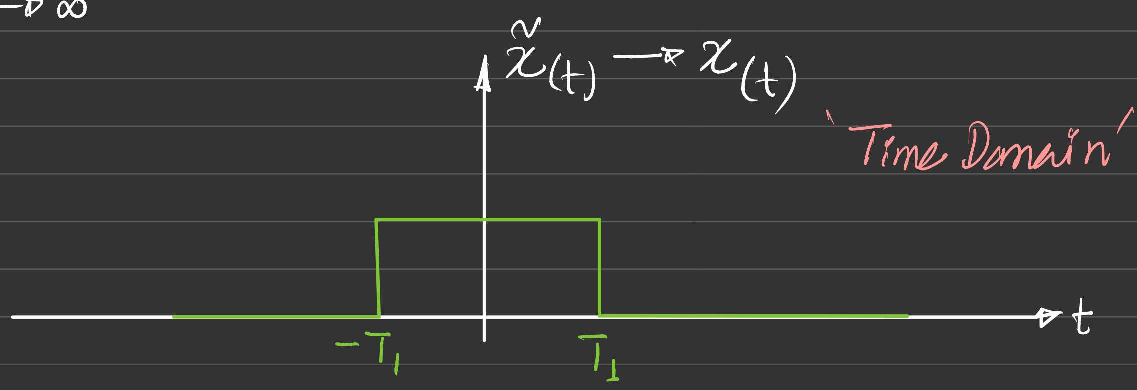
$$\therefore \tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0 \quad \text{— ④}$$

- Larger 'T' for above example will result:





- As $T \to \infty$

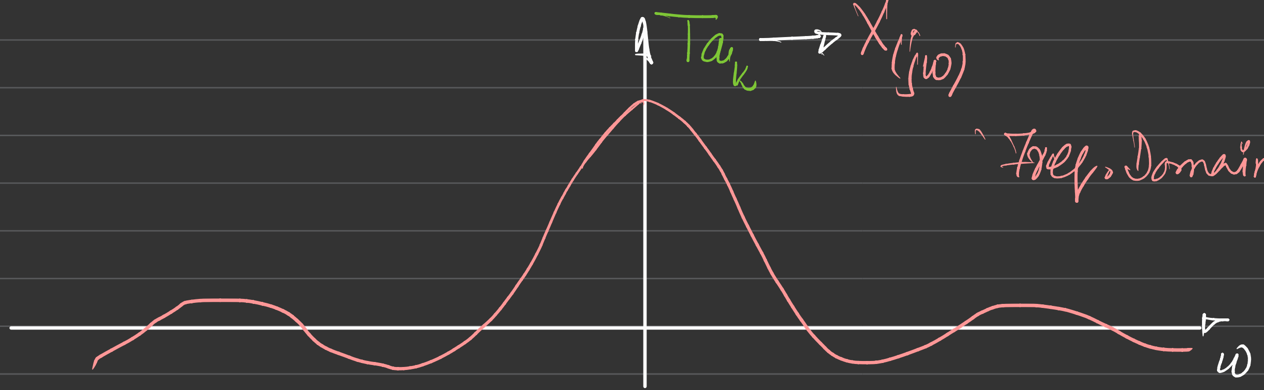


- And consequently, in the limit eqn ④ becomes a representation of $x_{(t)}$.

- Moreover, $\omega_0 \to 0$ as $T \to \infty$, & the RHS of eqn ④ passes to an integral.

$\therefore$ eqn ④ $\Rightarrow$ $$x_{(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{(j\omega)} e^{j\omega t} \, d\omega$$

& eqn ③ $\Rightarrow$ $$X_{(j\omega)} = \int_{-\infty}^{\infty} x_{(t)} e^{-j\omega t} \, dt$$



CONCLUSION

- In general, a sig. $x_{(t)}$ can be represented as:

$$x_{(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{(j\omega)} e^{j\omega t} \, d\omega$$ Synthesis (Inverse FT)

where

$$X_{(j\omega)} = \int_{-\infty}^{\infty} x_{(t)} e^{-j\omega t} \, dt$$ Analysis (FT of $x_{(t)}$)

- The transform $X_{(j\omega)}$ of an aperiodic sig. $x_{(t)}$ is commonly referred to as the spectrum of $x_{(t)}$.

- It provides us with the information needed for describing $x_{(t)}$ as a linear combination (specifically an integral) of sinusoidal signals at different frequencies.

NOTE: Convergence Conditions – Similar to CTFS

Example 1

$$x(t) = e^{-at}u(t) \quad , \quad a > 0 \quad \text{(Aperiodic)}$$

To represent it in freq. domain

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{-a-j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}$$

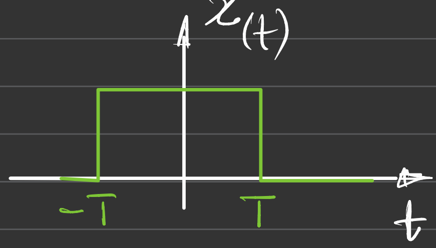
i.e.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a+j\omega} e^{j\omega t} d\omega$$

Example 2

Rectangular Pulse Signal (very useful)

$$x(t) = \begin{cases} 1 & , \ |t| < T \\ 0 & , \ |t| > T \end{cases}$$



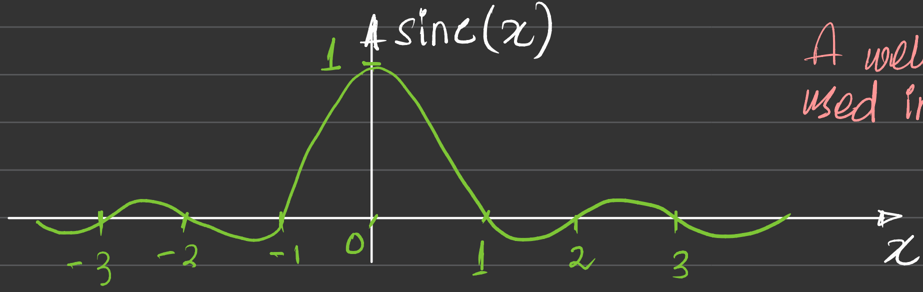
$$X(j\omega) = \int_{-T}^{T} e^{-j\omega t} dt$$

$$= \frac{1}{j\omega} e^{-j\omega t} \Big|_{-T}^{T} = \frac{2\sin(\omega T)}{\omega}$$

$$= 2T \frac{\sin(\omega T)}{\omega T} = 2T \,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right) \quad \text{—} (A)$$

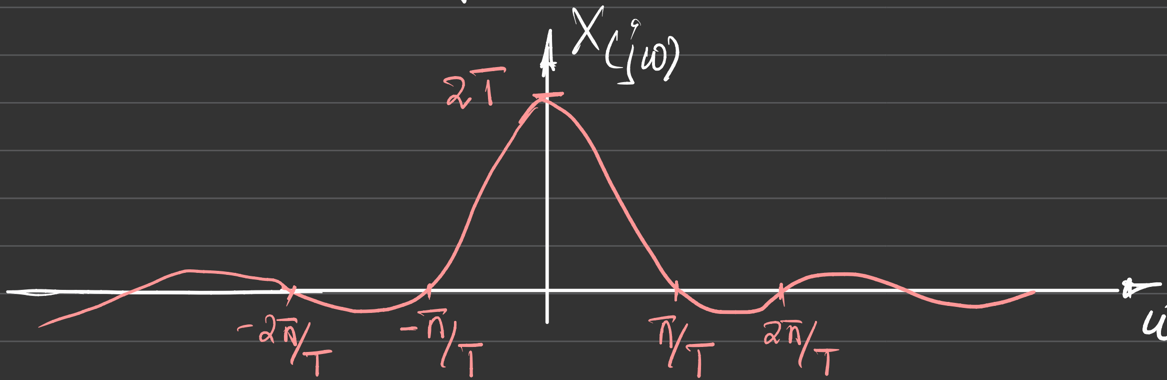
using:

Definition $\quad \mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$



A well known function used in Commn. sys.

- Accordingly, $X(j\omega)$ given in (A) is drawn as:



larger T, narrower spectrum band (RECIPROCAL SPREADING)